5.1 Perpendiculars and Bisectors

What you should learn

GOAL 1 Use properties of perpendicular bisectors.

GOAL 2 Use properties of angle bisectors to identify equal distances, such as the lengths of beams in a roof truss in **Example 3**.

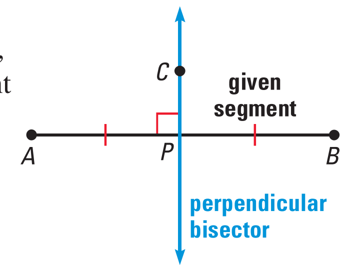
Why you should learn it

▼ To solve **real-life** problems, such as deciding where a hockey goalie should be positioned in **Exs. 33–35**.

REAL LIFE

GOAL 1 USING PROPERTIES OF PERPENDICULAR BISECTORS

In Lesson 1.5, you learned that a segment bisector intersects a segment at its midpoint. A segment, ray, line, or plane that is perpendicular to a segment at its midpoint is called a **perpendicular bisector**.



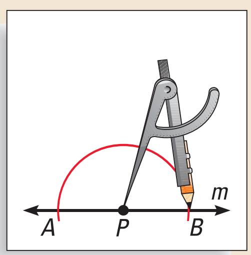
$\overleftrightarrow{CP}$ **is a $\perp$ bisector of $\overline{AB}$.**

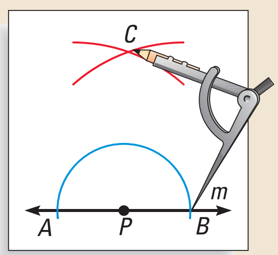
The construction below shows how to draw a line that is perpendicular to a given line or segment at a point P. You can use this method to construct a perpendicular bisector of a segment, as described below the activity.

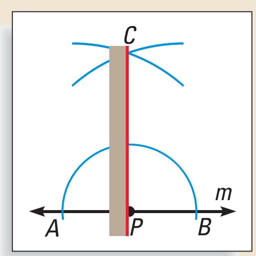
ACTIVITY
Construction

Perpendicular Through a Point on a Line

Use these steps to construct a line that is perpendicular to a given line m and that passes through a given point P on m.







1. Place the compass point at P. Draw an arc that intersects line m twice. Label the intersections as A and B.
2. Use a compass setting greater than AP. Draw an arc from A. With the same setting, draw an arc from B. Label the intersection of the arcs as C.
3. Use a straightedge to draw $\overleftrightarrow{CP}$. This line is perpendicular to line m and passes through P.

You can measure $\angle CPA$ on your construction to verify that the constructed line is perpendicular to the given line m. In the construction, $\overleftrightarrow{CP} \perp \overline{AB}$ and $PA = PB$, so $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

A point is **equidistant from two points** if its distance from each point is the same. In the construction above, C is equidistant from A and B because C was drawn so that $CA = CB$.

STUDENT HELP

Look Back
For a construction of a perpendicular to a line through a point not on the given line, see p. 130.

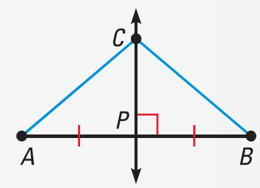
Theorem 5.1 below states that *any* point on the perpendicular bisector $\overleftrightarrow{CP}$ in the construction is equidistant from A and B, the endpoints of the segment. The converse helps you prove that a given point lies on a perpendicular bisector.

THEOREMS

THEOREM 5.1 ***Perpendicular Bisector Theorem***

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

If $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$, then $CA = CB$.

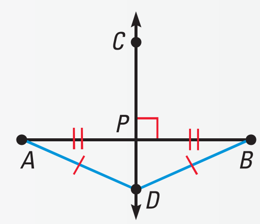


$CA = CB$

THEOREM 5.2 ***Converse of the Perpendicular Bisector Theorem***

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

If $DA = DB$, then D lies on the perpendicular bisector of $\overline{AB}$.



D is on $\overleftrightarrow{CP}$.

Proof

Plan for Proof of Theorem 5.1 Refer to the diagram for Theorem 5.1 above. Suppose that you are given that $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$. Show that right triangles $\triangle APC$ and $\triangle BPC$ are congruent using the SAS Congruence Postulate. Then show that $\overline{CA} \cong \overline{CB}$.

Exercise 28 asks you to write a two-column proof of Theorem 5.1 using this plan for proof. Exercise 29 asks you to write a proof of Theorem 5.2.

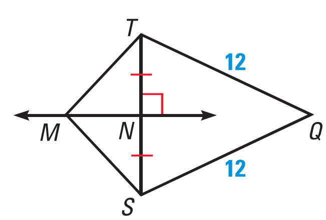
1 *Using Perpendicular Bisectors*

Logical Reasoning

In the diagram shown, $\overleftrightarrow{MN}$ is the perpendicular bisector of $\overline{ST}$.

a. What segment lengths in the diagram are equal?

b. Explain why Q is on $\overleftrightarrow{MN}$.



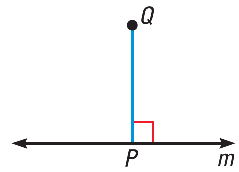
SOLUTION

a. $\overleftrightarrow{MN}$ bisects $\overline{ST}$, so $NS = NT$. Because M is on the perpendicular bisector of $\overline{ST}$, $MS = MT$ (by Theorem 5.1). The diagram shows that $QS = QT = 12$.

b. $QS = QT$, so Q is equidistant from S and T. By Theorem 5.2, Q is on the perpendicular bisector of $\overline{ST}$, which is $\overleftrightarrow{MN}$.

GOAL 2 USING PROPERTIES OF ANGLE BISECTORS

The **distance from a point to a line** is defined as the length of the perpendicular segment from the point to the line. For instance, in the diagram shown, the distance between the point Q and the line m is QP.



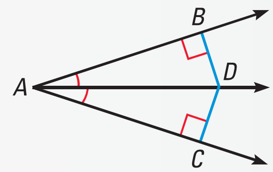
When a point is the same distance from one line as it is from another line, then the point is **equidistant from the two lines** (or rays or segments). The theorems below show that a point in the interior of an angle is equidistant from the sides of the angle if and only if the point is on the bisector of the angle.

THEOREMS

THEOREM 5.3 ***Angle Bisector Theorem***

If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

If $m\angle BAD = m\angle CAD$, then $DB = DC$.

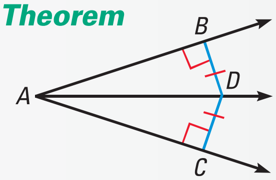


$DB = DC$

THEOREM 5.4 ***Converse of the Angle Bisector Theorem***

If a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the bisector of the angle.

If $DB = DC$, then $m\angle BAD = m\angle CAD$.



$m\angle BAD = m\angle CAD$

A paragraph proof of Theorem 5.3 is given in Example 2. Exercise 32 asks you to write a proof of Theorem 5.4.

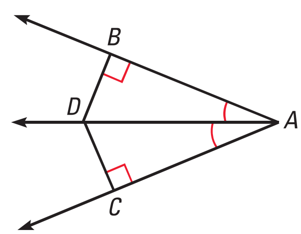
EXAMPLE 2 *Proof of Theorem 5.3*

Proof

GIVEN ▶ D is on the bisector of $\angle BAC$.
$\overline{DB} \perp \overrightarrow{AB}$, $\overline{DC} \perp \overrightarrow{AC}$

PROVE ▶ $DB = DC$

Plan for Proof Prove that $\triangle ADB \cong \triangle ADC$. Then conclude that $\overline{DB} \cong \overline{DC}$, so $DB = DC$.



SOLUTION

Paragraph Proof By the definition of an angle bisector, $\angle BAD \cong \angle CAD$. Because $\angle ABD$ and $\angle ACD$ are right angles, $\angle ABD \cong \angle ACD$. By the Reflexive Property of Congruence, $\overline{AD} \cong \overline{AD}$. Then $\triangle ADB \cong \triangle ADC$ by the AAS Congruence Theorem. Because corresponding parts of congruent triangles are congruent, $\overline{DB} \cong \overline{DC}$. By the definition of congruent segments, $DB = DC$.

FOCUS ON CAREERS

REAL LIFE

ENGINEERING TECHNICIAN

In manufacturing, engineering technicians prepare specifications for products such as roof trusses, and devise and run tests for quality control.

INTERNET CAREER LINK
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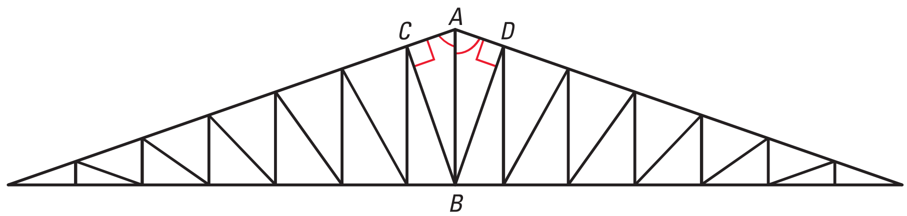
3 *Using Angle Bisectors*

ROOF TRUSSES Some roofs are built with wooden trusses that are assembled in a factory and shipped to the building site. In the diagram of the roof truss shown below, you are given that $\overrightarrow{AB}$ bisects $\angle CAD$ and that $\angle ACB$ and $\angle ADB$ are right angles. What can you say about $\overline{BC}$ and $\overline{BD}$?



SOLUTION

Because $\overline{BC}$ and $\overline{BD}$ meet $\overline{AC}$ and $\overline{AD}$ at right angles, they are perpendicular segments to the sides of $\angle CAD$. This implies that their lengths represent the distances from the point B to $\overrightarrow{AC}$ and $\overrightarrow{AD}$. Because point B is on the bisector of $\angle CAD$, it is equidistant from the sides of the angle.

▶ So, $BC = BD$, and you can conclude that $\overline{BC} \cong \overline{BD}$.

GUIDED PRACTICE

Vocabulary Check ✓

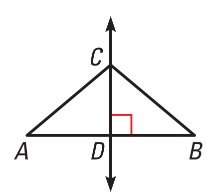
1. If D is on the __?__ of $\overline{AB}$, then D is *equidistant* from A and B.

Concept Check ✓

2. Point G is in the interior of $\angle HJK$ and is equidistant from the sides of the angle, $\overrightarrow{JH}$ and $\overrightarrow{JK}$. What can you conclude about G? Use a sketch to support your answer.

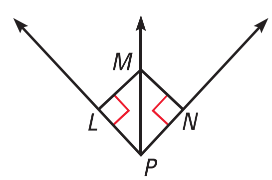
Skill Check ✓

In the diagram, $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.



3. What is the relationship between $\overline{AD}$ and $\overline{BD}$?
4. What is the relationship between $\angle ADC$ and $\angle BDC$?
5. What is the relationship between $\overline{AC}$ and $\overline{BC}$? Explain your answer.

In the diagram, $\overrightarrow{PM}$ is the bisector of $\angle LPN$.



6. What is the relationship between $\angle LPM$ and $\angle NPM$?
7. How is the distance between point M and $\overrightarrow{PL}$ related to the distance between point M and $\overrightarrow{PN}$?

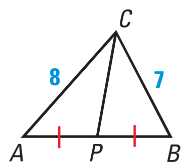
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 811.

LOGICAL REASONING **Tell whether the information in the diagram allows you to conclude that *C* is on the perpendicular bisector of $\overline{AB}$. Explain your reasoning.**

8.



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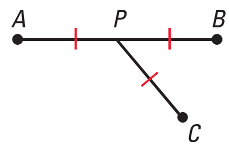

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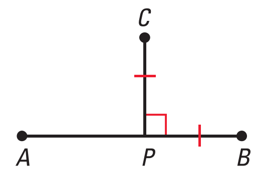
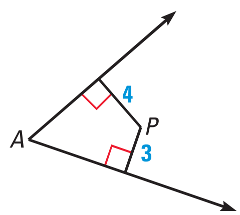


LOGICAL REASONING **In Exercises 11–13, tell whether the information in the diagram allows you to conclude that *P* is on the bisector of $\angle A$. Explain.**

11.



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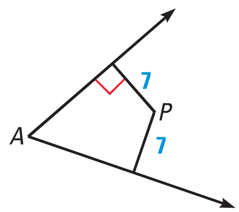
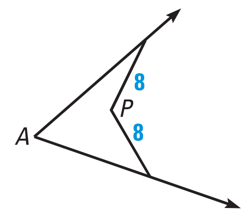


13.



14. **CONSTRUCTION** Draw $\overline{AB}$ with a length of 8 centimeters. Construct a perpendicular bisector and draw a point D on the bisector so that the distance between D and $\overline{AB}$ is 3 centimeters. Measure $\overline{AD}$ and $\overline{BD}$.

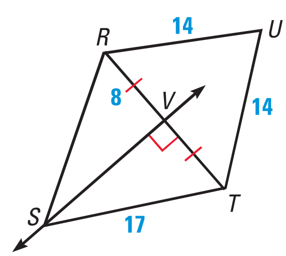
15. **CONSTRUCTION** Draw a large $\angle A$ with a measure of 60°. Construct the angle bisector and draw a point D on the bisector so that $AD = 3$ inches. Draw perpendicular segments from D to the sides of $\angle A$. Measure these segments to find the distance between D and the sides of $\angle A$.

USING PERPENDICULAR BISECTORS **Use the diagram shown.**

16. In the diagram, $\overleftrightarrow{SV} \perp \overline{RT}$ and $\overline{VR} \cong \overline{VT}$. Find VT.

17. In the diagram, $\overleftrightarrow{SV} \perp \overline{RT}$ and $\overline{VR} \cong \overline{VT}$. Find SR.

18. In the diagram, $\overleftrightarrow{SV}$ is the perpendicular bisector of $\overline{RT}$. Because $UR = UT = 14$, what can you conclude about point U?



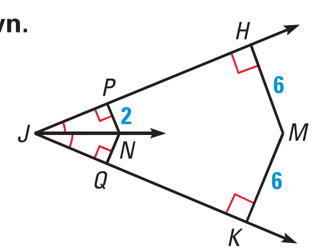
STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 8–10, 14, 16–18, 21–26
Example 2: Exs. 11–13, 15, 19, 20, 21–26
Example 3: Exs. 31, 33–35

USING ANGLE BISECTORS **Use the diagram shown.**

19. In the diagram, $\overrightarrow{JN}$ bisects $\angle HJK$, $\overline{NP} \perp \overrightarrow{JP}$, $\overline{NQ} \perp \overrightarrow{JQ}$, and $NP = 2$. Find NQ.

20. In the diagram, $\overrightarrow{JN}$ bisects $\angle HJK$, $\overline{MH} \perp \overrightarrow{JH}$, $\overline{MK} \perp \overrightarrow{JK}$, and $MH = MK = 6$. What can you conclude about point M?

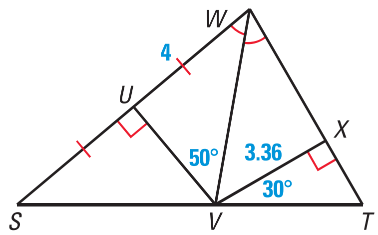


USING BISECTOR THEOREMS **In Exercises 21–26, match the angle measure or segment length described with its correct value.**

A. 60° **B.** 8

C. 40° **D.** 4

E. 50° **F.** 3.36



21. SW **22.** $m\angle XTV$

23. $m\angle VWX$ **24.** VU

25. WX **26.** $m\angle WVX$

27. **PROVING A CONSTRUCTION** Write a proof to verify that $\overleftrightarrow{CP} \perp \overline{AB}$ in the construction on page 264.

STUDENT HELP

Look Back
For help with proving that constructions are valid, see p. 231.

28. **PROVING THEOREM 5.1** Write a proof of Theorem 5.1, the Perpendicular Bisector Theorem. You may want to use the plan for proof given on page 265.

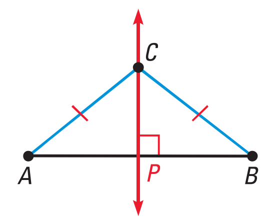
GIVEN ▶ $\overleftrightarrow{CP}$ is the perpendicular bisector of $\overline{AB}$.

PROVE ▶ C is equidistant from A and B.

29. **PROVING THEOREM 5.2** Use the diagram shown to write a two-column proof of Theorem 5.2, the Converse of the Perpendicular Bisector Theorem.

GIVEN ▶ C is equidistant from A and B.

PROVE ▶ C is on the perpendicular bisector of $\overline{AB}$.

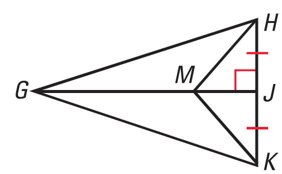


Plan for Proof Use the Perpendicular Postulate to draw $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$. Show that $\triangle APC \cong \triangle BPC$ by the HL Congruence Theorem. Then $\overline{AP} \cong \overline{BP}$, so $AP = BP$.

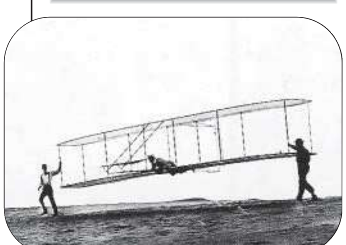
30. **PROOF** Use the diagram shown.

GIVEN ▶ $\overline{GJ}$ is the perpendicular bisector of $\overline{HK}$.

PROVE ▶ $\triangle GHM \cong \triangle GKM$

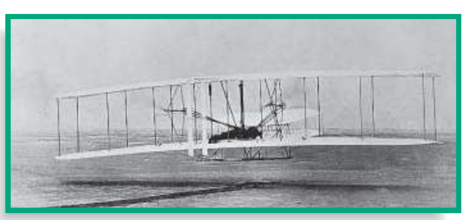


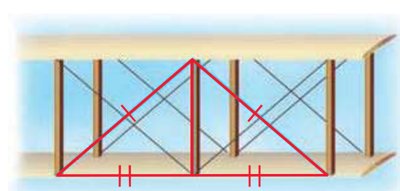
FOCUS ON PEOPLE



THE WRIGHT BROTHERS
In Kitty Hawk, North Carolina, on December 17, 1903, Orville and Wilbur Wright became the first people to successfully fly an engine-driven, heavier-than-air machine.

31. **EARLY AIRCRAFT** On many of the earliest airplanes, wires connected vertical posts to the edges of the wings, which were wooden frames covered with cloth. Suppose the lengths of the wires from the top of a post to the edges of the frame are the same and the distances from the bottom of the post to the ends of the two wires are the same. What does that tell you about the post and the section of frame between the ends of the wires?

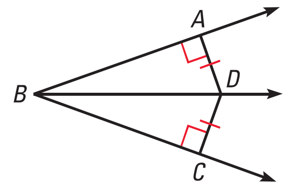




32. **DEVELOPING PROOF** Use the diagram to complete the proof of Theorem 5.4, the Converse of the Angle Bisector Theorem.

GIVEN ▶ D is in the interior of $\angle ABC$ and is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

PROVE ▶ D lies on the angle bisector of $\angle ABC$.



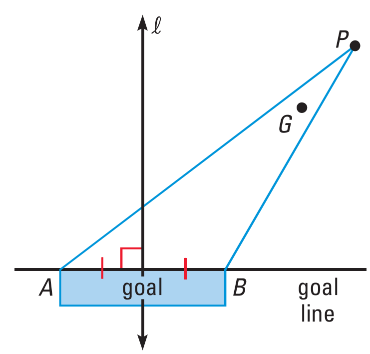
Statements	Reasons
1. D is in the interior of $\angle ABC$.	**1.** ___?___
2. D is ___?___ from $\overrightarrow{BA}$ and $\overrightarrow{BC}$.	**2.** Given
3. ___?___ = ___?___	**3.** Definition of equidistant
4. $\overline{DA} \perp$ ___?___, ___?___ $\perp \overrightarrow{BC}$	**4.** Definition of distance from a point to a line
5. ___?___	**5.** If 2 lines are $\perp$, then they form 4 rt. $\angle$s.
6. ___?___	**6.** Definition of right triangle
7. $\overline{BD} \cong \overline{BD}$	**7.** ___?___
8. ___?___	**8.** HL Congruence Thm.
9. $\angle ABD \cong \angle CBD$	**9.** ___?___
10. $\overrightarrow{BD}$ bisects $\angle ABC$ and point D is on the bisector of $\angle ABC$.	**10.** ___?___

ICE HOCKEY **In Exercises 33–35, use the following information.** In the diagram, the goalie is at point G and the puck is at point P. The goalie's job is to prevent the puck from entering the goal.

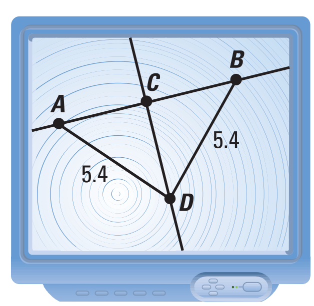
33. When the puck is at the other end of the rink, the goalie is likely to be standing on line ℓ. How is ℓ related to $\overline{AB}$?

34. As an opposing player with the puck skates toward the goal, the goalie is likely to move from line ℓ to other places on the ice. What should be the relationship between $\overrightarrow{PG}$ and $\angle APB$?

35. How does $m\angle APB$ change as the puck gets closer to the goal? Does this change make it easier or more difficult for the goalie to defend the goal? Explain.



36. **TECHNOLOGY** Use geometry software to construct $\overline{AB}$. Find the midpoint C. Draw the perpendicular bisector of $\overline{AB}$ through C. Construct a point D along the perpendicular bisector and measure $\overline{DA}$ and $\overline{DB}$. Move D along the perpendicular bisector. What theorem does this construction demonstrate?



Test Preparation

37. MULTI-STEP PROBLEM Use the map shown and the following information. A town planner is trying to decide whether a new household X should be covered by fire station A, B, or C.

a. Trace the map and draw the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

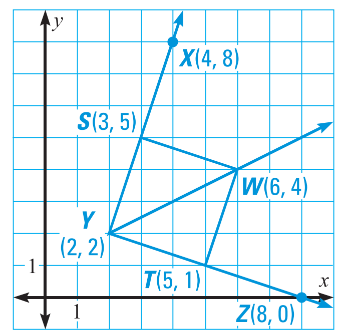
b. Construct the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Do the perpendicular bisectors meet at a point?

c. The perpendicular bisectors divide the town into regions. Shade the region closest to fire station A red. Shade the region closest to fire station B blue. Shade the region closest to fire station C gray.

d. *Writing* In an emergency at household X, which fire station should respond? Explain your choice.

★ Challenge

USING ALGEBRA **Use the graph at the right.**



38. Use slopes to show that $\overline{WS} \perp \overrightarrow{YX}$ and that $\overline{WT} \perp \overrightarrow{YZ}$.

39. Find WS and WT.

40. Explain how you know that $\overrightarrow{YW}$ bisects $\angle XYZ$.

EXTRA CHALLENGE
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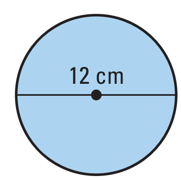
MIXED REVIEW

CIRCLES **Find the missing measurement for the circle shown. Use 3.14 as an approximation for π.** **(Review 1.7 for 5.2)**



41. radius **42.** circumference **43.** area

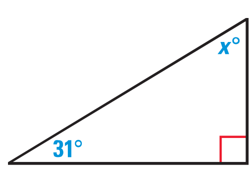
CALCULATING SLOPE **Find the slope of the line that passes through the given points.** **(Review 3.6)**

44. $A(-1, 5)$, $B(-2, 10)$ **45.** $C(4, -3)$, $D(-6, 5)$ **46.** $E(4, 5)$, $F(9, 5)$

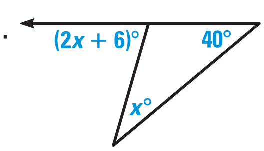
47. $G(0, 8)$, $H(-7, 0)$ **48.** $J(3, 11)$, $K(-10, 12)$ **49.** $L(-3, -8)$, $M(8, -8)$

USING ALGEBRA **Find the value of x.** **(Review 4.1)**

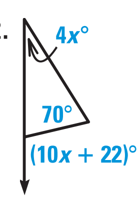
50.



51.



52.



5.2 Bisectors of a Triangle

What *you should learn*

GOAL 1 Use properties of perpendicular bisectors of a triangle, as applied in **Example 1**.

GOAL 2 Use properties of angle bisectors of a triangle.

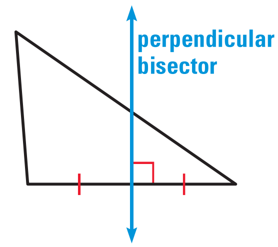
Why *you should learn it*

▼ To solve **real-life** problems, such as finding the center of a mushroom ring in **Exs. 24–26**.

REAL LIFE

GOAL 1 USING PERPENDICULAR BISECTORS OF A TRIANGLE

In Lesson 5.1, you studied properties of perpendicular bisectors of segments and angle bisectors. In this lesson, you will study the special cases in which the segments and angles being bisected are parts of a triangle.



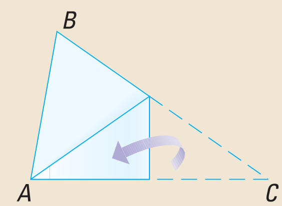
A **perpendicular bisector of a triangle** is a line (or ray or segment) that is perpendicular to a side of the triangle at the midpoint of the side.

ACTIVITY
Developing Concepts

Perpendicular Bisectors of a Triangle

1. Cut four large acute scalene triangles out of paper. Make each one different.

2. Choose one triangle. Fold the triangle to form the perpendicular bisectors of the sides. Do the three bisectors intersect at the same point?

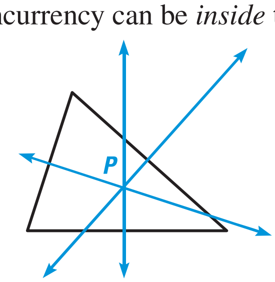


3. Repeat the process for the other three triangles. What do you observe? Write your observation in the form of a conjecture.

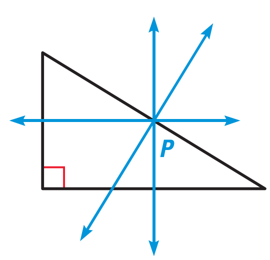
4. Choose one triangle. Label the vertices A, B, and C. Label the point of intersection of the perpendicular bisectors as P. Measure $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$. What do you observe?

When three or more lines (or rays or segments) intersect in the same point, they are called **concurrent lines** (or rays or segments). The point of intersection of the lines is called the **point of concurrency**.

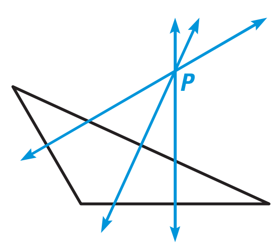
The three perpendicular bisectors of a triangle are concurrent. The point of concurrency can be *inside* the triangle, *on* the triangle, or *outside* the triangle.



acute triangle



right triangle



obtuse triangle

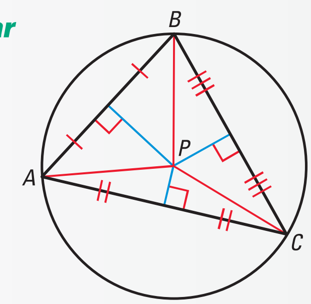
The point of concurrency of the perpendicular bisectors of a triangle is called the **circumcenter of the triangle.** In each triangle at the bottom of page 272, the circumcenter is at *P*. The circumcenter of a triangle has a special property, as described in Theorem 5.5. You will use coordinate geometry to illustrate this theorem in Exercises 29–31. A proof appears on page 835.

THEOREM

THEOREM 5.5 ***Concurrency of Perpendicular Bisectors of a Triangle***

The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

$$PA = PB = PC$$

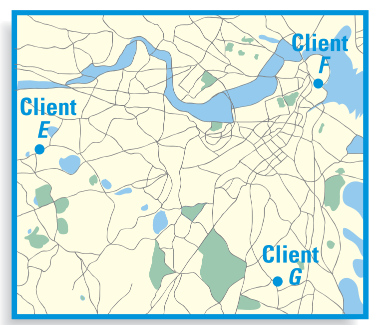


The diagram for Theorem 5.5 shows that the circumcenter is the center of the circle that passes through the vertices of the triangle. The circle is *circumscribed* about $\triangle ABC$. Thus, the radius of this circle is the distance from the center to any of the vertices.

1 Using Perpendicular Bisectors

REAL LIFE

FACILITIES PLANNING A company plans to build a distribution center that is convenient to three of its major clients. The planners start by roughly locating the three clients on a sketch and finding the circumcenter of the triangle formed.



a. Explain why using the circumcenter as the location of a distribution center would be convenient for all the clients.

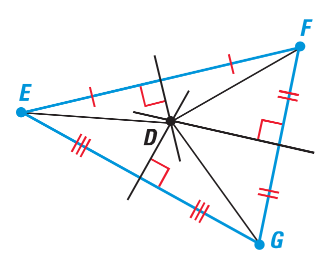
b. Make a sketch of the triangle formed by the clients. Locate the circumcenter of the triangle. Tell what segments are congruent.

SOLUTION

a. Because the circumcenter is equidistant from the three vertices, each client would be equally close to the distribution center.

b. Label the vertices of the triangle as *E*, *F*, and *G*. Draw the perpendicular bisectors. Label their intersection as *D*.

▶ By Theorem 5.5, $DE = DF = DG$.



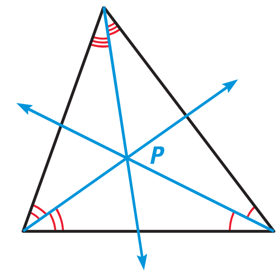
STUDENT HELP

INTERNET

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING ANGLE BISECTORS OF A TRIANGLE

An **angle bisector of a triangle** is a bisector of an angle of the triangle. The three angle bisectors are concurrent. The point of concurrency of the angle bisectors is called the **incenter of the triangle**, and it always lies inside the triangle. The incenter has a special property that is described below in Theorem 5.6. Exercise 22 asks you to write a proof of this theorem.

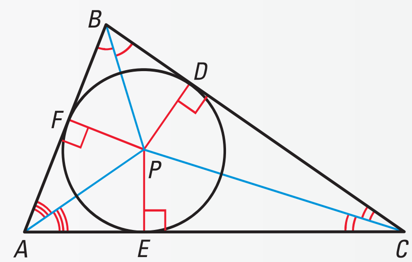


THEOREM

THEOREM 5.6 ***Concurrency of Angle Bisectors of a Triangle***

The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

$$PD = PE = PF$$



The diagram for Theorem 5.6 shows that the incenter is the center of the circle that touches each side of the triangle once. The circle is *inscribed* within $\triangle ABC$. Thus, the radius of this circle is the distance from the center to any of the sides.

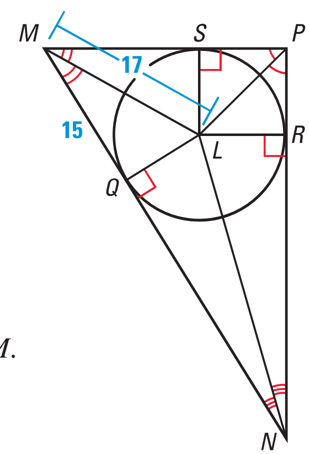
EXAMPLE 2 *Using Angle Bisectors*

Logical Reasoning

The angle bisectors of $\triangle MNP$ meet at point L.

a. What segments are congruent?

b. Find LQ and LR.



SOLUTION

a. By Theorem 5.6, the three angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle. So, $\overline{LR} \cong \overline{LQ} \cong \overline{LS}$.

b. Use the Pythagorean Theorem to find LQ in $\triangle LQM$.

$$(LQ)^2 + (MQ)^2 = (LM)^2$$

$$(LQ)^2 + 15^2 = 17^2 \quad \text{Substitute.}$$

$$(LQ)^2 + 225 = 289 \quad \text{Multiply.}$$

$$(LQ)^2 = 64 \quad \text{Subtract 225 from each side.}$$

$$LQ = 8 \quad \text{Find the positive square root.}$$

▶ So, $LQ = 8$ units. Because $\overline{LR} \cong \overline{LQ}$, $LR = 8$ units.

STUDENT HELP

Look Back
For help with the Pythagorean Theorem, see p. 20.

GUIDED PRACTICE

Vocabulary Check ✓

1. If three or more lines intersect at the same point, the lines are __?__.

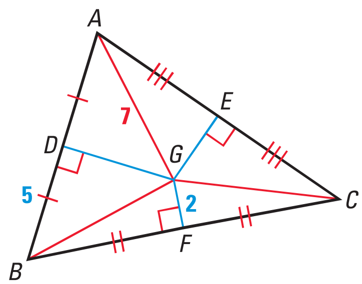
Concept Check ✓

2. Think of something about the words *incenter* and *circumcenter* that you can use to remember which special parts of a triangle meet at each point.

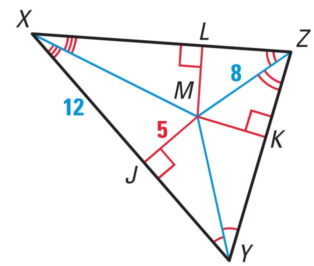
Skill Check ✓

Use the diagram and the given information to find the indicated measure.

3. The perpendicular bisectors of $\triangle ABC$ meet at point G. Find GC.



4. The angle bisectors of $\triangle XYZ$ meet at point M. Find MK.



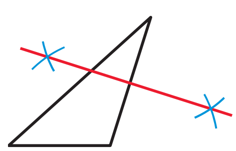
PRACTICE AND APPLICATIONS

STUDENT HELP

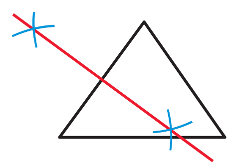
Extra Practice
to help you master skills is on p. 811.

CONSTRUCTION **Draw a large example of the given type of triangle. Construct perpendicular bisectors of the sides. (See page 264.) For the type of triangle, do the bisectors intersect *inside, on,* or *outside* the triangle?**

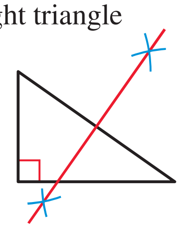
5. obtuse triangle



6. acute triangle



7. right triangle



DRAWING CONCLUSIONS **Draw a large $\triangle ABC$.**

8. Construct the angle bisectors of $\triangle ABC$. Label the point where the angle bisectors meet as D.

9. Construct perpendicular segments from D to each of the sides of the triangle. Measure each segment. What do you notice? Which theorem have you just confirmed?

LOGICAL REASONING **Use the results of Exercises 5–9 to complete the statement using *always, sometimes,* or *never.***

10. A perpendicular bisector of a triangle __?__ passes through the midpoint of a side of the triangle.

11. The angle bisectors of a triangle __?__ intersect at a single point.

12. The angle bisectors of a triangle __?__ meet at a point outside the triangle.

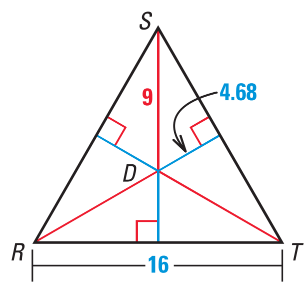
13. The circumcenter of a triangle __?__ lies outside the triangle.

STUDENT HELP

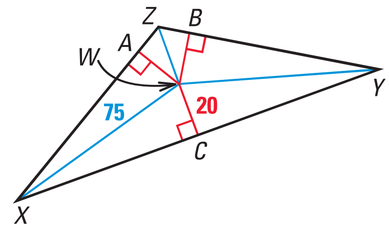
HOMEWORK HELP
Example 1: Exs. 5–7, 10–13, 14, 17, 20, 21
Example 2: Exs. 8, 9, 10–13, 15, 16, 22

BISECTORS In each case, find the indicated measure.

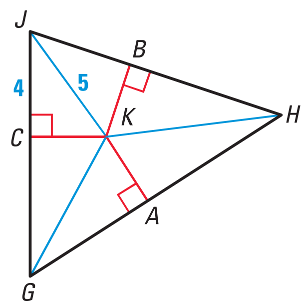
14. The perpendicular bisectors of $\triangle RST$ meet at point D. Find DR.



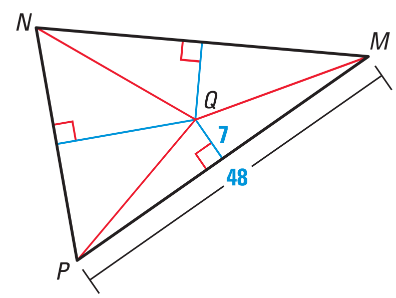
15. The angle bisectors of $\triangle XYZ$ meet at point W. Find WB.



16. The angle bisectors of $\triangle GHJ$ meet at point K. Find KB.

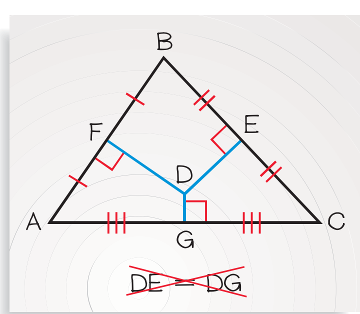


17. The perpendicular bisectors of $\triangle MNP$ meet at point Q. Find QN.

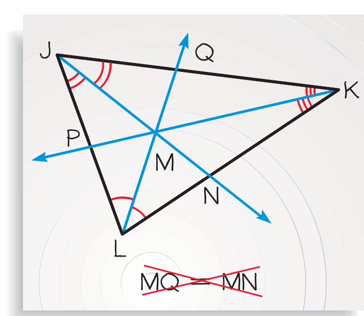


ERROR ANALYSIS Explain why the student's conclusion is *false*. Then state a correct conclusion that can be deduced from the diagram.

18.

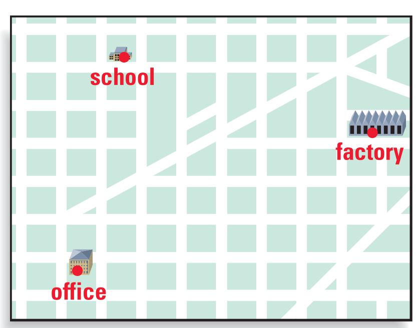


19.



LOGICAL REASONING In Exercises 20 and 21, use the following information and map.

Your family is considering moving to a new home. The diagram shows the locations of where your parents work and where you go to school. The locations form a triangle.



20. In the diagram, how could you find a point that is equidistant from each location? Explain your answer.

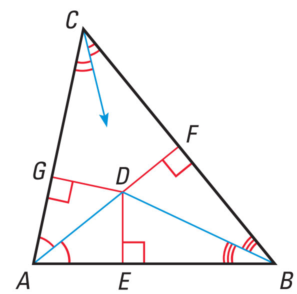
21. Make a sketch of the situation. Find the best location for the new home.

22. **DEVELOPING PROOF** Complete the proof of Theorem 5.6, the Concurrency of Angle Bisectors.

GIVEN ▶ $\triangle ABC$, the bisectors of $\angle A$, $\angle B$, and $\angle C$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, $\overline{DG} \perp \overline{CA}$

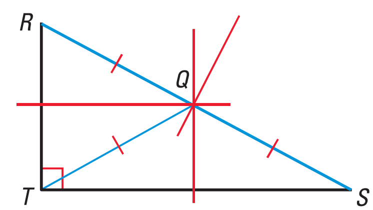
PROVE ▶ The angle bisectors intersect at a point that is equidistant from $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

Plan for Proof Show that D, the point of intersection of the bisectors of $\angle A$ and $\angle B$, also lies on the bisector of $\angle C$. Then show that D is equidistant from the sides of the triangle.



Statements	Reasons
1. $\triangle ABC$, the bisectors of $\angle A$, $\angle B$, and $\angle C$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{BC}$, $\overline{DG} \perp \overline{CA}$	**1.** Given
2. $\underline{\ \ ?\ \ } = DG$	**2.** $\overrightarrow{AD}$ bisects $\angle BAC$, so D is $\underline{\ \ ?\ \ }$ from the sides of $\angle BAC$.
3. $DE = DF$	**3.** $\underline{\ \ ?\ \ }$
4. $DF = DG$	**4.** $\underline{\ \ ?\ \ }$
5. D is on the $\underline{\ \ ?\ \ }$ of $\angle C$.	**5.** Converse of the Angle Bisector Theorem
6. $\underline{\ \ ?\ \ }$	**6.** Givens and Steps $\underline{\ \ ?\ \ }$

23. *Writing* Joannie thinks that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. Explain how she could use perpendicular bisectors to verify her conjecture.

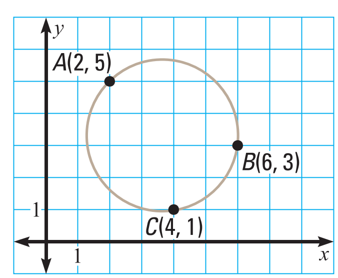


FOCUS ON APPLICATIONS

MUSHROOMS live for only a few days. As the mycelium spreads outward, new mushroom rings are formed. A mushroom ring in France is almost half a mile in diameter and is about 700 years old.

SCIENCE CONNECTION **In Exercises 24–26, use the following information.**
A *mycelium* fungus grows underground in all directions from a central point. Under certain conditions, mushrooms sprout up in a ring at the edge. The radius of the mushroom ring is an indication of the mycelium's age.

24. Suppose three mushrooms in a mushroom ring are located as shown. Make a large copy of the diagram and draw $\triangle ABC$. Each unit on your coordinate grid should represent 1 foot.



25. Draw perpendicular bisectors on your diagram to find the center of the mushroom ring. Estimate the radius of the ring.

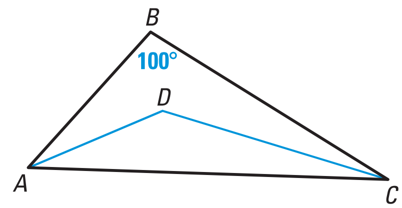
26. Suppose the radius of the mycelium increases at a rate of about 8 inches per year. Estimate its age.

Test Preparation

MULTIPLE CHOICE **Choose the correct answer from the list given.**

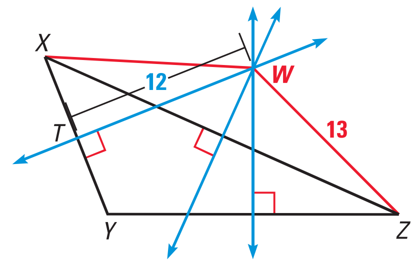
27. $\overline{AD}$ and $\overline{CD}$ are angle bisectors of $\triangle ABC$ and $m\angle ABC = 100°$. Find $m\angle ADC$.

(A) 80° (B) 90° (C) 100°
(D) 120° (E) 140°



28. The perpendicular bisectors of $\triangle XYZ$ intersect at point W, $WT = 12$, and $WZ = 13$. Find XY.

(A) 5 (B) 8 (C) 10
(D) 12 (E) 13



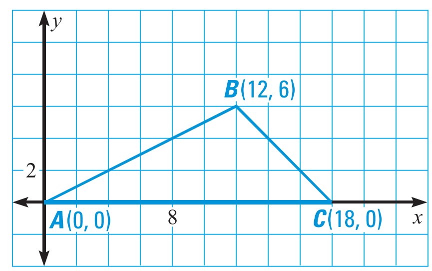
★ Challenge

USING ALGEBRA **Use the graph of $\triangle ABC$ to illustrate Theorem 5.5, the Concurrency of Perpendicular Bisectors.**

29. Find the midpoint of each side of $\triangle ABC$. Use the midpoints to find the equations of the perpendicular bisectors of $\triangle ABC$.

30. Using your equations from Exercise 29, find the intersection of two of the lines. Show that the point is on the third line.

31. Show that the point in Exercise 30 is equidistant from the vertices of $\triangle ABC$.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

FINDING AREAS **Find the area of the triangle described.** (Review 1.7 for 5.3)

32. base = 9, height = 5 **33.** base = 22, height = 7

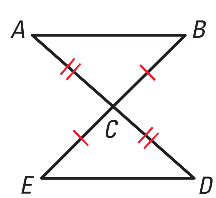
WRITING EQUATIONS **The line with the given equation is perpendicular to line *j* at point *P*. Write an equation of line *j*.** (Review 3.7)

34. $y = 3x - 2,\ P(1, 4)$ **35.** $y = -2x + 5,\ P(7, 6)$

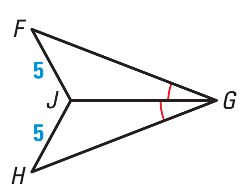
36. $y = -\frac{2}{3}x - 1,\ P(2, 8)$ **37.** $y = \frac{10}{11}x + 3,\ P(-2, -9)$

LOGICAL REASONING **Decide whether enough information is given to prove that the triangles are congruent. If there is enough information, tell which congruence postulate or theorem you would use.** (Review 4.3, 4.4, and 4.6)

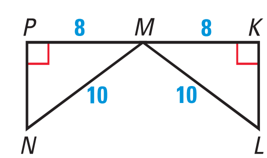
38.



39.



40.



5.3 Medians and Altitudes of a Triangle

***What** you should learn*

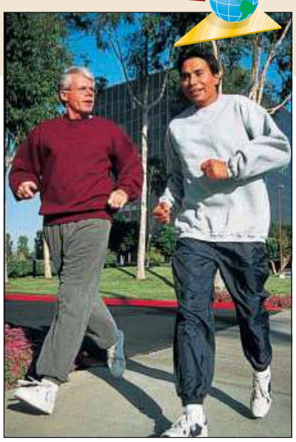
GOAL 1 Use properties of medians of a triangle.

GOAL 2 Use properties of altitudes of a triangle.

***Why** you should learn it*

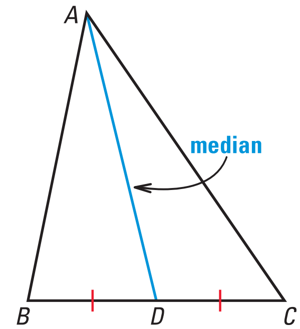
▼ To solve **real-life** problems, such as locating points in a triangle used to measure a person's heart fitness as in **Exs. 30–33**.

REAL LIFE



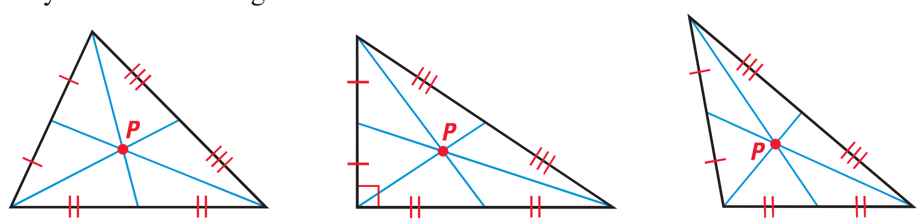
GOAL 1 USING MEDIANS OF A TRIANGLE

In Lesson 5.2, you studied two special types of segments of a triangle: perpendicular bisectors of the sides and angle bisectors. In this lesson, you will study two other special types of segments of a triangle: *medians* and *altitudes*.



A **median of a triangle** is a segment whose endpoints are a vertex of the triangle and the midpoint of the opposite side. For instance, in $\triangle ABC$ shown at the right, D is the midpoint of side $\overline{BC}$. So, $\overline{AD}$ is a median of the triangle.

The three medians of a triangle are concurrent. The point of concurrency is called the **centroid of the triangle**. The centroid, labeled P in the diagrams below, is always inside the triangle.



acute triangle **right triangle** **obtuse triangle**

The medians of a triangle have a special concurrency property, as described in Theorem 5.7. Exercises 13–16 ask you to use paper folding to demonstrate the relationships in this theorem. A proof appears on pages 836–837.

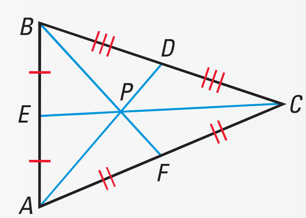
THEOREM

THEOREM 5.7 ***Concurrency of Medians of a Triangle***

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

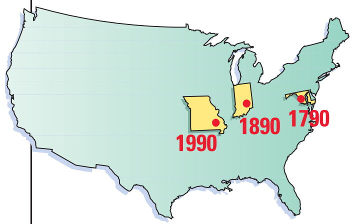
If P is the centroid of $\triangle ABC$, then

$AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BF$, and $CP = \frac{2}{3}CE$.



The centroid of a triangle can be used as its balancing point, as shown on the next page.

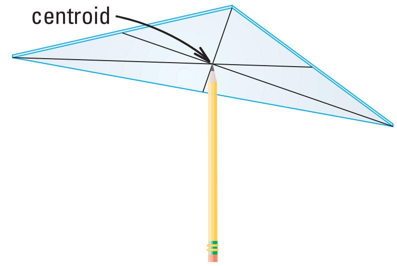
FOCUS ON APPLICATIONS



REAL LIFE

CENTER OF POPULATION
Suppose the location of each person counted in a census is identified by a weight placed on a flat, weightless map of the United States. The map would balance at a point that is the center of the population. This center has been moving westward over time.

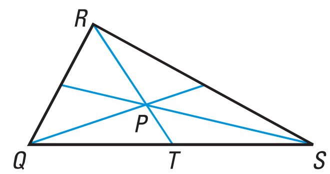
A triangular model of uniform thickness and density will balance at the centroid of the triangle. For instance, in the diagram shown at the right, the triangular model will balance if the tip of a pencil is placed at its centroid.



1 *Using the Centroid of a Triangle*

P is the centroid of $\triangle QRS$ shown below and $PT = 5$. Find RT and RP.

SOLUTION



Because P is the centroid, $RP = \frac{2}{3}RT$.

Then $PT = RT - RP = \frac{1}{3}RT$.

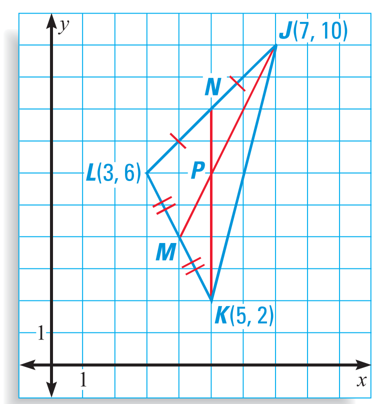
Substituting 5 for PT, $5 = \frac{1}{3}RT$, so $RT = 15$.

Then $RP = \frac{2}{3}RT = \frac{2}{3}(15) = 10$.

▶ So, $RP = 10$ and $RT = 15$.

2 *Finding the Centroid of a Triangle*

Find the coordinates of the centroid of $\triangle JKL$.



SOLUTION

You know that the centroid is two thirds of the distance from each vertex to the midpoint of the opposite side.

Choose the median $\overline{KN}$. Find the coordinates of N, the midpoint of $\overline{JL}$. The coordinates of N are

$$\left(\frac{3+7}{2}, \frac{6+10}{2}\right) = \left(\frac{10}{2}, \frac{16}{2}\right) = (5, 8).$$

Find the distance from vertex K to midpoint N. The distance from $K(5, 2)$ to $N(5, 8)$ is $8 - 2$, or 6 units.

Determine the coordinates of the centroid, which is $\frac{2}{3} \cdot 6$, or **4** units up from vertex K along the median $\overline{KN}$.

▶ The coordinates of centroid P are $(5, 2 + \mathbf{4})$, or $(5, 6)$.

. . . .

Exercises 21–23 ask you to use the Distance Formula to confirm that the distance from vertex J to the centroid P in Example 2 is two thirds of the distance from J to M, the midpoint of the opposite side.

STUDENT HELP

INTERNET

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

GOAL 2 USING ALTITUDES OF A TRIANGLE

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side. An altitude can lie inside, on, or outside the triangle.

Every triangle has three altitudes. The lines containing the altitudes are concurrent and intersect at a point called the **orthocenter of the triangle**.

Logical Reasoning

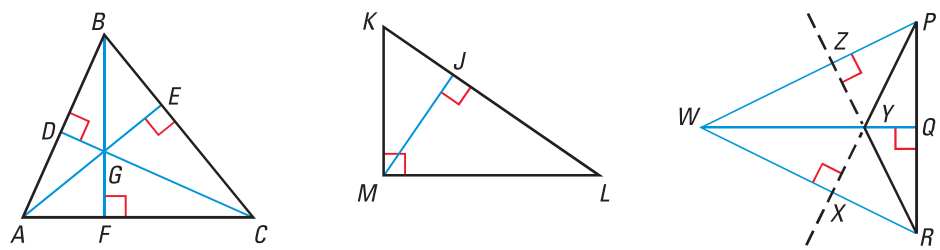
EXAMPLE 3 *Drawing Altitudes and Orthocenters*

Where is the orthocenter located in each type of triangle?

a. Acute triangle **b.** Right triangle **c.** Obtuse triangle

SOLUTION

Draw an example of each type of triangle and locate its orthocenter.



a. $\triangle ABC$ is an acute triangle. The three altitudes intersect at G, a point *inside* the triangle.

b. $\triangle KLM$ is a right triangle. The two legs, $\overline{LM}$ and $\overline{KM}$, are also altitudes. They intersect at the triangle's right angle. This implies that the orthocenter is *on* the triangle at M, the vertex of the right angle of the triangle.

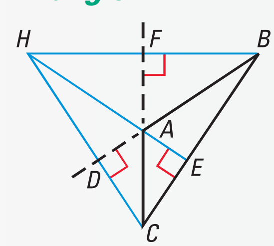
c. $\triangle YPR$ is an obtuse triangle. The three lines that contain the altitudes intersect at W, a point that is *outside* the triangle.

THEOREM

THEOREM 5.8 ***Concurrency of Altitudes of a Triangle***

The lines containing the altitudes of a triangle are concurrent.

If $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are the altitudes of $\triangle ABC$, then the lines $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, and $\overleftrightarrow{CD}$ intersect at some point H.



Exercises 24–26 ask you to use construction to verify Theorem 5.8. A proof appears on page 838.

GUIDED PRACTICE

Vocabulary Check ✓

1. The *centroid* of a triangle is the point where the three __?__ intersect.

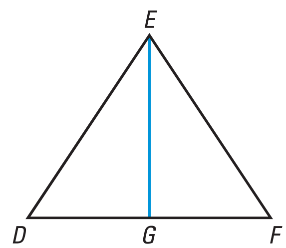
Concept Check ✓

2. In Example 3 on page 281, explain why the two legs of the right triangle in part (b) are also altitudes of the triangle.

Skill Check ✓

Use the diagram shown and the given information to decide in each case whether $\overline{EG}$ is a *perpendicular bisector*, an *angle bisector*, a *median*, or an *altitude* of $\triangle DEF$.

3. $\overline{DG} \cong \overline{FG}$
4. $\overline{EG} \perp \overline{DF}$
5. $\angle DEG \cong \angle FEG$
6. $\overline{EG} \perp \overline{DF}$ and $\overline{DG} \cong \overline{FG}$
7. $\triangle DGE \cong \triangle FGE$



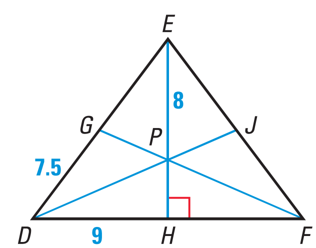
PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice
to help you master skills is on p. 811.

USING MEDIANS OF A TRIANGLE **In Exercises 8–12, use the figure below and the given information.**

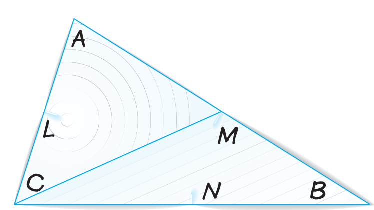
P is the centroid of $\triangle DEF$, $\overline{EH} \perp \overline{DF}$,
$DH = 9$, $DG = 7.5$, $EP = 8$, and $DE = FE$.



8. Find the length of $\overline{FH}$.
9. Find the length of $\overline{EH}$.
10. Find the length of $\overline{PH}$.
11. Find the perimeter of $\triangle DEF$.
12. **LOGICAL REASONING** In the diagram of $\triangle DEF$ above, $\frac{EP}{EH} = \frac{2}{3}$.

    Find $\frac{PH}{EH}$ and $\frac{PH}{EP}$.

PAPER FOLDING **Cut out a large acute, right, or obtuse triangle. Label the vertices. Follow the steps in Exercises 13–16 to verify Theorem 5.7.**

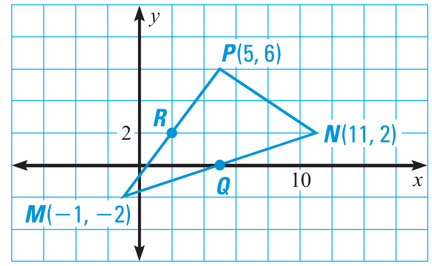


13. Fold the sides to locate the midpoint of each side. Label the midpoints.
14. Fold to form the median from each vertex to the midpoint of the opposite side.
15. Did your medians meet at about the same point? If so, label this centroid point.
16. Verify that the distance from the centroid to a vertex is two thirds of the distance from that vertex to the midpoint of the opposite side.

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 8–11, 13–16
Example 2: Exs. 17–23
Example 3: Exs. 24–26

USING ALGEBRA Use the graph shown.



17. Find the coordinates of Q, the midpoint of $\overline{MN}$.

18. Find the length of the median $\overline{PQ}$.

19. Find the coordinates of the centroid. Label this point as T.

20. Find the coordinates of R, the midpoint of $\overline{MP}$. Show that the quotient $\frac{NT}{NR}$ is $\frac{2}{3}$.

USING ALGEBRA Refer back to Example 2 on page 280.

21. Find the coordinates of M, the midpoint of $\overline{KL}$.

22. Use the Distance Formula to find the lengths of $\overline{JP}$ and $\overline{JM}$.

23. Verify that $JP = \frac{2}{3}JM$.

STUDENT HELP

Look Back
To construct an altitude, use the construction of a perpendicular to a line through a point not on the line, as shown on p. 130.

CONSTRUCTION Draw and label a large scalene triangle of the given type and construct the altitudes. Verify Theorem 5.8 by showing that the lines containing the altitudes are concurrent, and label the orthocenter.

24. an acute $\triangle ABC$

25. a right $\triangle EFG$ with right angle at G

26. an obtuse $\triangle KLM$

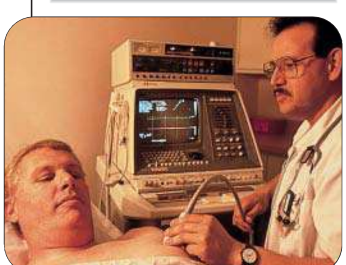
TECHNOLOGY Use geometry software to draw a triangle. Label the vertices as *A*, *B*, and *C*.

27. Construct the altitudes of $\triangle ABC$ by drawing perpendicular lines through each side to the opposite vertex. Label them $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

28. Find and label G and H, the intersections of $\overline{AD}$ and $\overline{BE}$ and of $\overline{BE}$ and $\overline{CF}$.

29. Prove that the altitudes are concurrent by showing that $GH = 0$.

FOCUS ON CAREERS

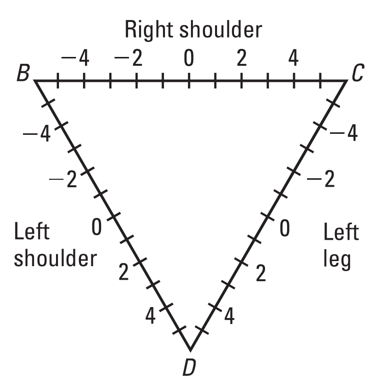


REAL LIFE

CARDIOLOGY TECHNICIAN
Technicians use equipment like electrocardiographs to test, monitor, and evaluate heart function.

INTERNET CAREER LINK
www.mcdougallittell.com

ELECTROCARDIOGRAPH In Exercises 30–33, use the following information about electrocardiographs.
The equilateral triangle $\triangle BCD$ is used to plot electrocardiograph readings. Consider a person who has a left shoulder reading (S) of -1, a right shoulder reading (R) of 2, and a left leg reading (L) of 3.



30. On a large copy of $\triangle BCD$, plot the reading to form the vertices of $\triangle SRL$. (This triangle is an *Einthoven's Triangle*, named for the inventor of the electrocardiograph.)

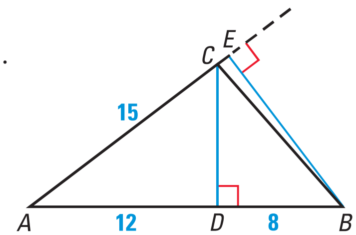
31. Construct the circumcenter M of $\triangle SRL$.

32. Construct the centroid P of $\triangle SRL$. Draw line r through P parallel to $\overline{BC}$.

33. Estimate the measure of the acute angle between line r and $\overline{MP}$. Cardiologists call this the angle of a person's heart.

Test Preparation

34. MULTI-STEP PROBLEM Recall the formula for the area of a triangle, $A = \frac{1}{2}bh$, where b is the length of the base and h is the height. The height of a triangle is the length of an altitude.



a. Make a sketch of $\triangle ABC$. Find CD, the height of the triangle (the length of the altitude to side $\overline{AB}$).

b. Use CD and AB to find the area of $\triangle ABC$.

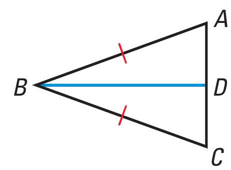
c. Draw $\overline{BE}$, the altitude to the line containing side $\overline{AC}$.

d. Use the results of part (b) to find the length of $\overline{BE}$.

e. *Writing* Write a formula for the length of an altitude in terms of the base and the area of the triangle. Explain.

★ Challenge

SPECIAL TRIANGLES **Use the diagram at the right.**



35. GIVEN ▶ $\triangle ABC$ is isosceles.
$\overline{BD}$ is a median to base $\overline{AC}$.

PROVE ▶ $\overline{BD}$ is also an altitude.

36. Are the medians to the *legs* of an isosceles triangle also altitudes? Explain your reasoning.

37. Are the medians of an *equilateral* triangle also altitudes? Are they contained in the angle bisectors? Are they contained in the perpendicular bisectors?

EXTRA CHALLENGE
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38. LOGICAL REASONING In a proof, if you are given a median of an equilateral triangle, what else can you conclude about the segment?

MIXED REVIEW

USING ALGEBRA **Write an equation of the line that passes through point *P* and is parallel to the line with the given equation.** (Review 3.6 for 5.4)

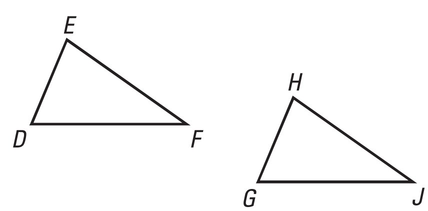
39. $P(1, 7)$, $y = -x + 3$

40. $P(-3, -8)$, $y = -2x - 3$

41. $P(4, -9)$, $y = 3x + 5$

42. $P(4, -2)$, $y = -\frac{1}{2}x - 1$

DEVELOPING PROOF **In Exercises 43 and 44, state the third congruence that must be given to prove that $\triangle DEF \cong \triangle GHJ$ using the indicated postulate or theorem.** (Review 4.4)



43. GIVEN ▶ $\angle D \cong \angle G$, $\overline{DF} \cong \overline{GJ}$
AAS Congruence Theorem

44. GIVEN ▶ $\angle E \cong \angle H$, $\overline{EF} \cong \overline{HJ}$
ASA Congruence Postulate

45. USING THE DISTANCE FORMULA Place a right triangle with legs of length 9 units and 13 units in a coordinate plane and use the Distance Formula to find the length of the hypotenuse. (Review 4.7)

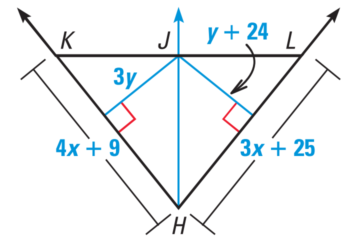
QUIZ 1

Self-Test for Lessons 5.1–5.3

Use the diagram shown and the given information. (Lesson 5.1)

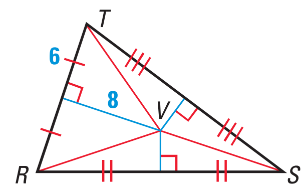
$\overline{HJ}$ is the perpendicular bisector of $\overline{KL}$.

$\overrightarrow{HJ}$ bisects $\angle KHL$.

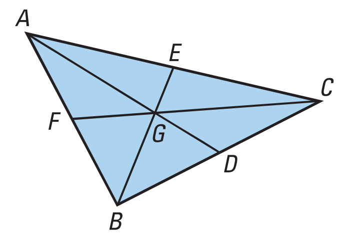


1. Find the value of x.
2. Find the value of y.

In the diagram shown, the perpendicular bisectors of $\triangle RST$ meet at V. (Lesson 5.2)



3. Find the length of $\overline{VT}$.
4. What is the length of $\overline{VS}$? Explain.
5. **BUILDING A MOBILE** Suppose you want to attach the items in a mobile so that they hang horizontally. You would want to find the balancing point of each item. For the triangular metal plate shown, describe where the balancing point would be located. (Lesson 5.3)



$\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians. $CF = 12$ in.

MATH & History

Optimization

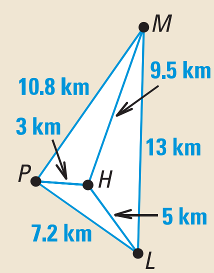
APPLICATION LINK
www.mcdougallittell.com

THEN

THROUGHOUT HISTORY, people have faced problems involving minimizing resources or maximizing output, a process called optimization. The use of mathematics in solving these types of problems has increased greatly since World War II, when mathematicians found the optimal shape for naval convoys to avoid enemy fire.

NOW

TODAY, with the help of computers, optimization techniques are used in many industries, including manufacturing, economics, and architecture.



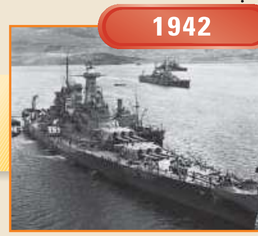
1. Your house is located at point H in the diagram. You need to do errands at the post office (P), the market (M), and the library (L). In what order should you do your errands to minimize the distance traveled?
2. Look back at Exercise 34 on page 270. Explain why the goalie's position on the angle bisector optimizes the chances of blocking a scoring shot.

1611

Johannes Kepler proposes the optimal way to stack cannonballs.

WWII naval convoy

1942



1972

This Olympic stadium roof uses a minimum of materials.

Thomas Hales proves Kepler's cannonball conjecture.

1997